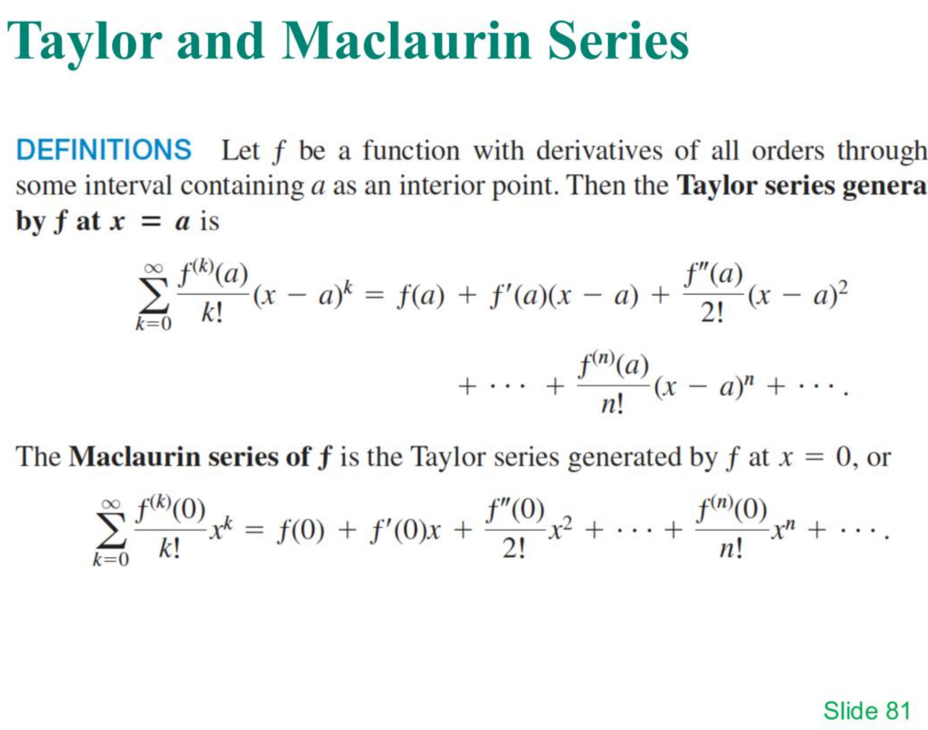
Section 10.7

Taylor and Maclaurin Series



Taylor and Maclaurin Series

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots .$$

Page 2 Annotation

Title: Definitions of Taylor Series and Maclaurin Series

Clear Explanation:

The **Taylor series** of a function f centered at a point a is an infinite series made from the function’s value and all its derivatives at x = a.

The **Maclaurin series** is simply a Taylor series when the center point a = 0.

The general formula is:

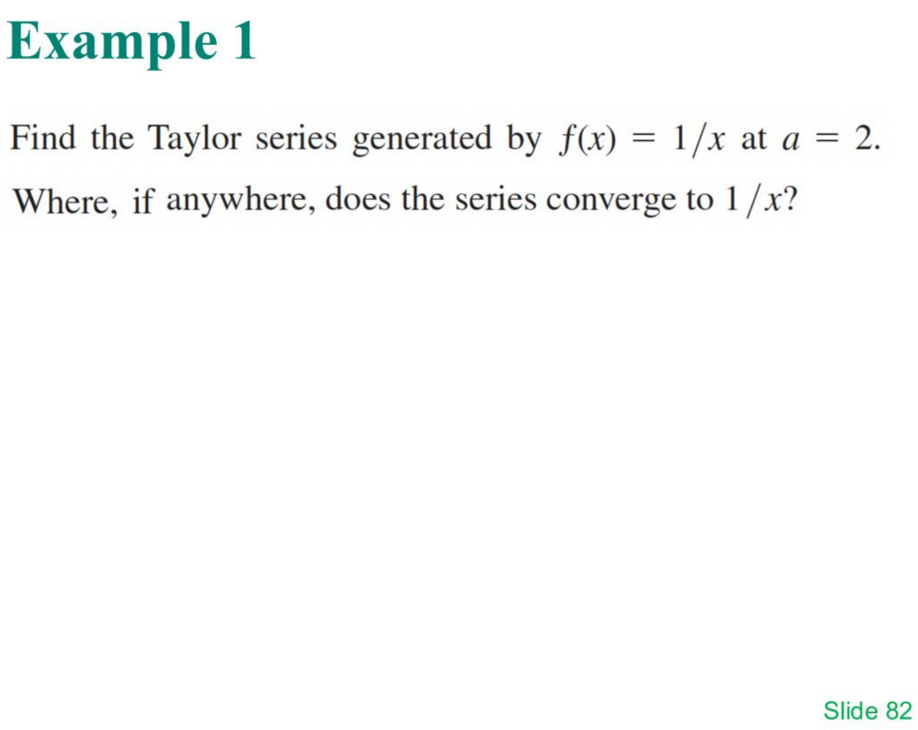
$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k$$

Key Points:

- Uses $f(a)$, $f'(a)$, $f''(a)$, $f'''(a)$, …
- Each term is divided by k! (factorial)
- Maclaurin series = Taylor series at a = 0

Why It Matters:

This formula is the foundation of the entire section. Almost every example in this chapter uses this definition.



Example 1

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

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Page 3 Annotation

Title: Example 1

Clear Explanation:

Find the Taylor series for the function f(x) = 1/x centered at a = 2.

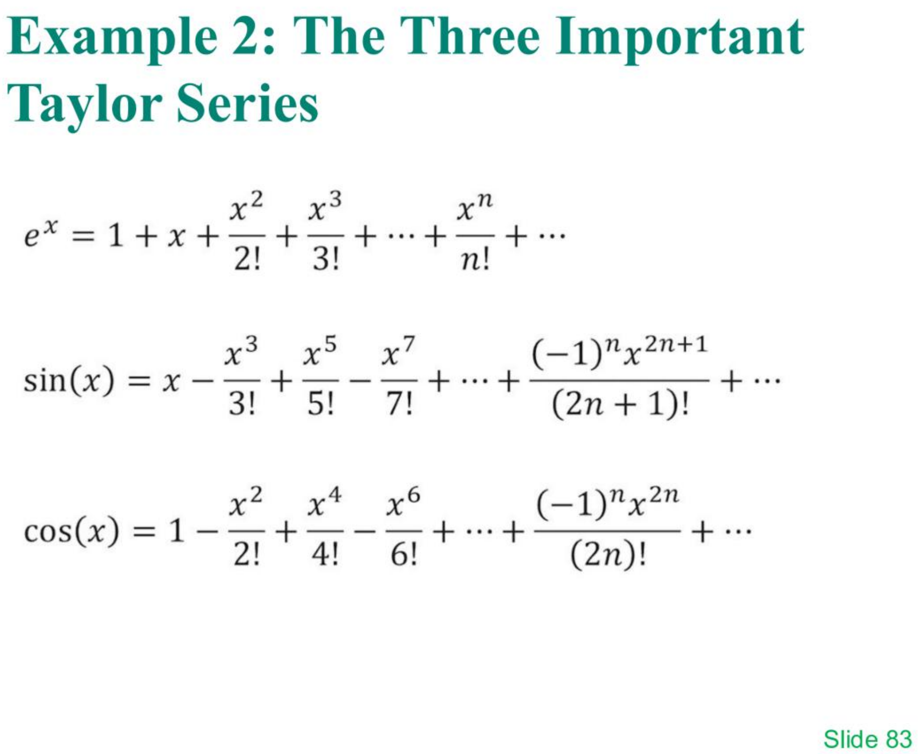
Then determine where (in what interval) this series actually equals the original function 1/x.

Key Points:

- Center point a = 2
- We need to compute all derivatives of 1/x at x = 2
- Test the interval of convergence

Why It Matters:

This is a classic example showing how to build a Taylor series for a simple rational function and how to check where the series is valid.



Example 2: The Three Important Taylor Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

Page 4 Annotation

Title: Example 2 – The Three Important Taylor Series

Clear Explanation:

These are the three most famous and useful Taylor (Maclaurin) series you should memorize:

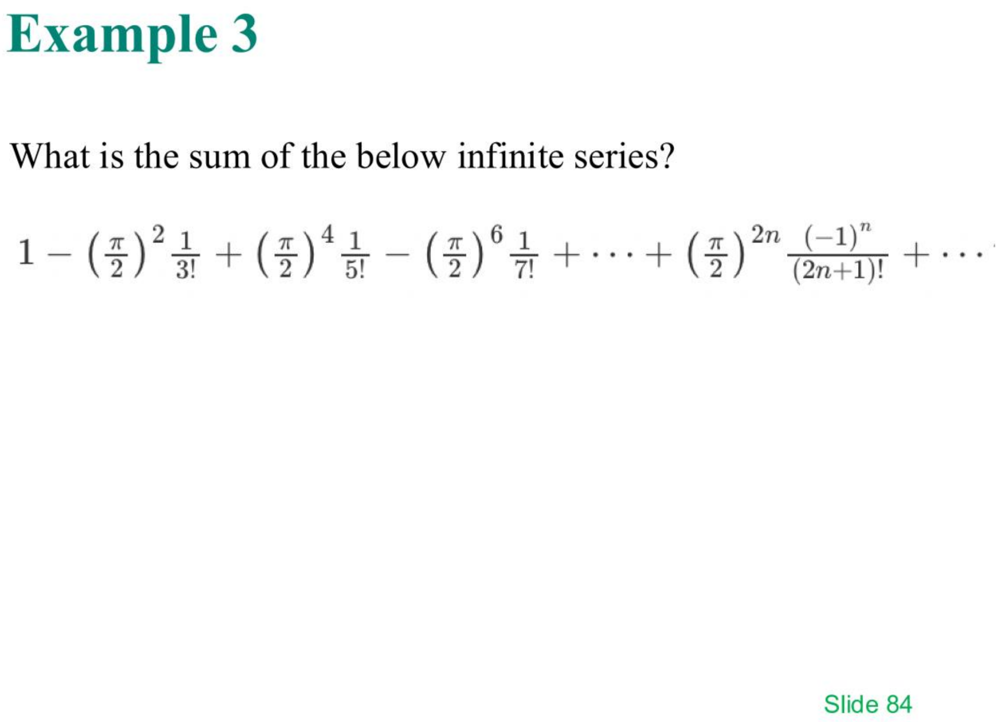
- e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots
- \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
- \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots

Key Points:

- All centered at 0 (Maclaurin series)
- They converge for all real x
- Used everywhere in calculus and physics

Why It Matters:

These three series are the “building blocks.” Many harder functions can be built by combining or substituting into these.



Example 3

What is the sum of the below infinite series?

$$1-\left(\frac{\pi}{2}\right)^{2}\frac{1}{3!}+\left(\frac{\pi}{2}\right)^{4}\frac{1}{5!}-\left(\frac{\pi}{2}\right)^{6}\frac{1}{7!}+\cdots+\left(\frac{\pi}{2}\right)^{2n}\frac{(-1)^{n}}{(2n+1)!}+\cdots$$

Page 5 Annotation

Title: Example 3

Clear Explanation:

What is the sum of this infinite series?

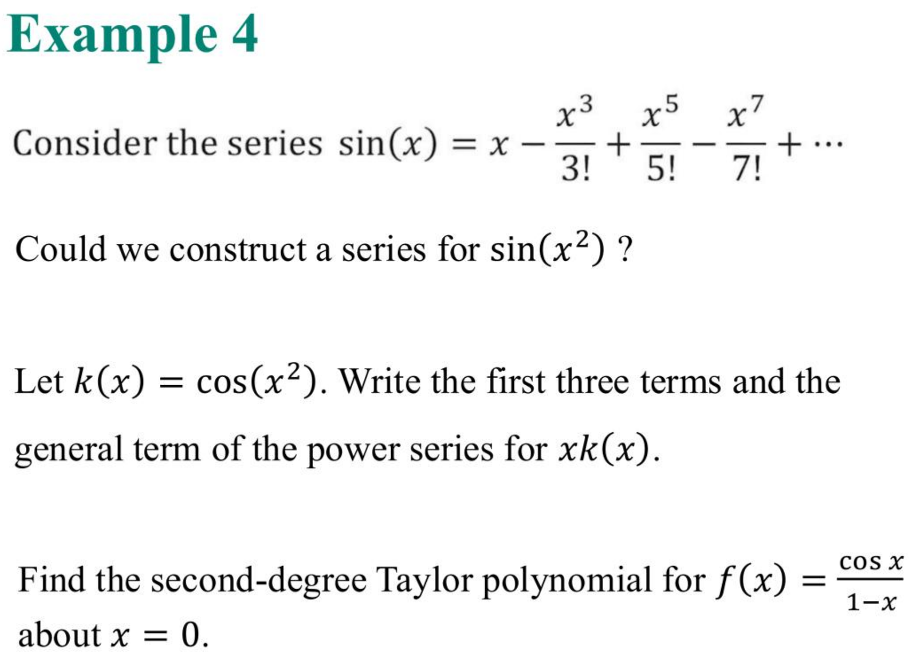
1 - \left(\frac{\pi}{2}\right)^2 \frac{1}{3!} + \left(\frac{\pi}{2}\right)^4 \frac{1}{5!} - \left(\frac{\pi}{2}\right)^6 \frac{1}{7!} + \cdots

Key Points:

- Recognize the pattern: it looks like the sine or cosine series with x = π/2
- The series is actually equal to sin(π/2) or cos(something)

Why It Matters:

This teaches you how to recognize known series inside a more complicated-looking expression and find its exact sum.



Example 4

Consider the series $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Could we construct a series for $\sin(x^2)$?

Let $k(x) = \cos(x^2)$. Write the first three terms and the general term of the power series for $xk(x)$.

Find the second-degree Taylor polynomial for $f(x) = \frac{\cos x}{1-x}$ about $x = 0$.

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Title: Example 4

Clear Explanation:

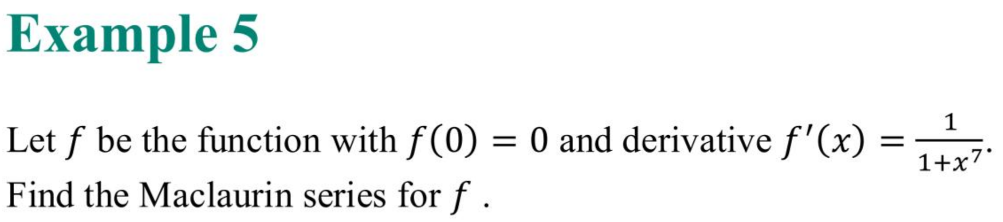
1. Can we build a series for $\sin(x^2)$?
2. Let $k(x) = \cos(x^2)$. Write the first three terms and the general term for the power series of $x \cdot k(x)$.
3. Find the second-degree Taylor polynomial for $f(x) = \cos x / (1 - x)$ about $x = 0$.

Key Points:

- Use substitution into known series (e.g., replace x with x^2 in sin x)
- Multiply series together
- Find Taylor polynomial of specific degree (here degree 2)

Why It Matters:

This shows real techniques for creating new series from the three basic ones you already know.



Example 5

Let f be the function with $f(0) = 0$ and derivative $f'(x) = \frac{1}{1+x^7}$.
Find the Maclaurin series for f .

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Page 7 Annotation

Title: Example 5

Clear Explanation:

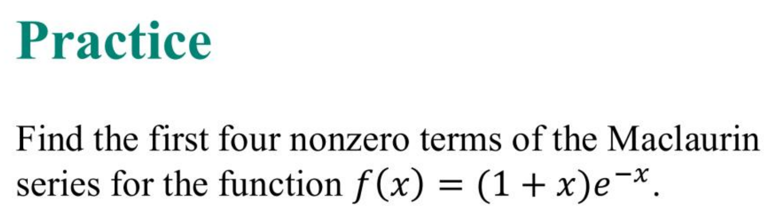
Let f be the function with $f(0) = 0$ and $f'(x) = 1/(1 + x^7)$.
Find the Maclaurin series for f.

Key Points:

- We are given the derivative, not the function itself
- Integrate the series for $f'(x)$ term by term to get the series for $f(x)$
- Don’t forget the constant of integration (here it is 0 because $f(0) = 0$)

Why It Matters:

This is a common technique: when you know the derivative series, integrate to get the original function’s series.



Practice

Find the first four nonzero terms of the Maclaurin series for the function $f(x) = (1 + x)e^{-x}$.

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Page 8 Annotation

Title: Practice Problem

Clear Explanation:

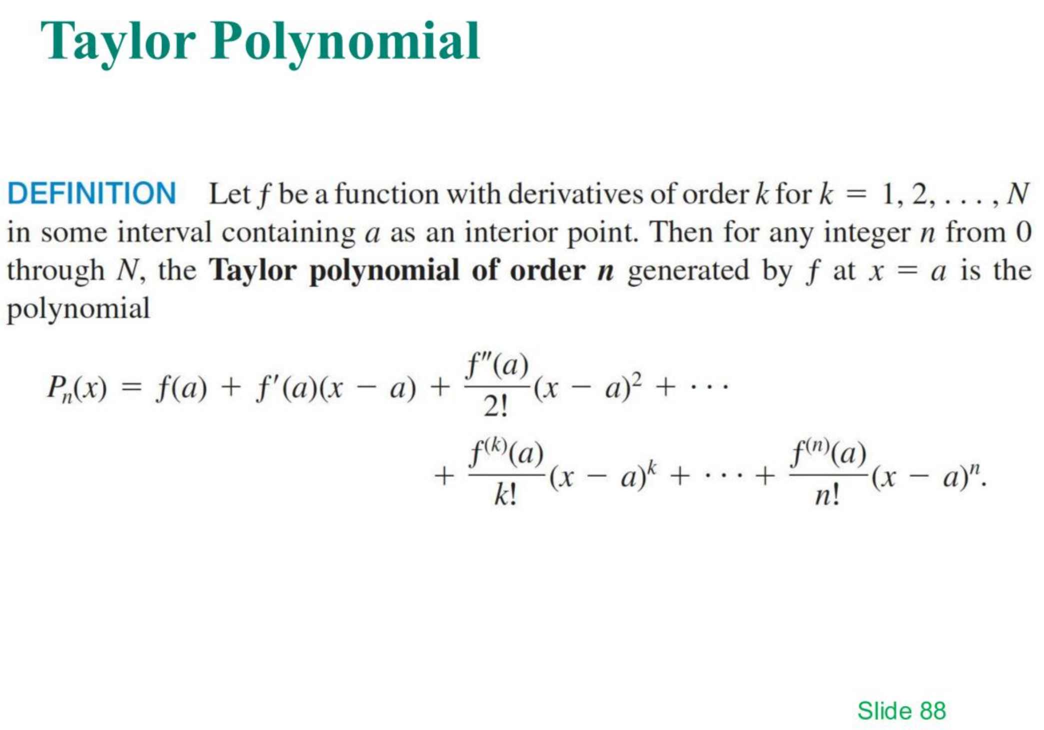
Find the first four nonzero terms of the Maclaurin series for the function
f(x) = (1 + x)e^{-x}.

Key Points:

- Use the product rule for series: multiply the series for (1 + x) and the series for e^{-x}
- Collect terms up to the first four nonzero terms

Why It Matters:

Excellent practice for combining two known series using multiplication.



Taylor Polynomial

DEFINITION Let f be a function with derivatives of order k for $k = 1, 2, \dots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

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Title: Taylor Polynomial (Definition)

Clear Explanation:

A **Taylor polynomial of order n** is the finite (truncated) version of the Taylor series.

It stops after the term with the n-th derivative.

Formula:

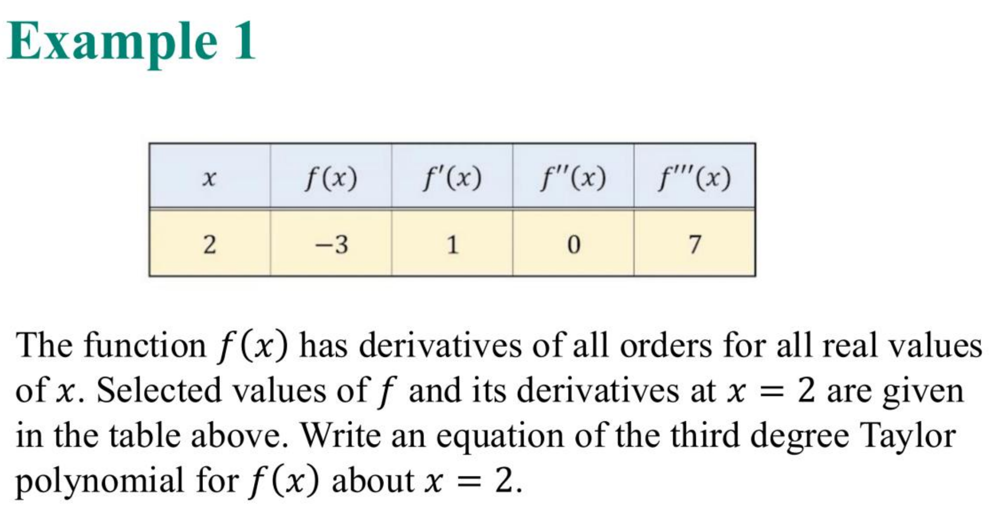
$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Key Points:

- Polynomial (finite sum)
- Order n means it goes up to the n-th derivative term
- Used for approximation when the infinite series is too hard to use

Why It Matters:

In real applications we usually cannot sum an infinite series, so we use a high-degree Taylor polynomial as a good approximation.



Example 1

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
2	-3	1	0	7

The function $f(x)$ has derivatives of all orders for all real values of x. Selected values of f and its derivatives at $x = 2$ are given in the table above. Write an equation of the third degree Taylor polynomial for $f(x)$ about $x = 2$.

Page 10 Annotation

Title: Example 1 (Table of derivatives)

Clear Explanation:

The function $f(x)$ has derivatives of all orders for all real x. A table gives selected values at $x = 2$:

$f(2) = -3$, $f'(2) = 1$, $f''(2) = 0$, $f'''(2) = 7$.

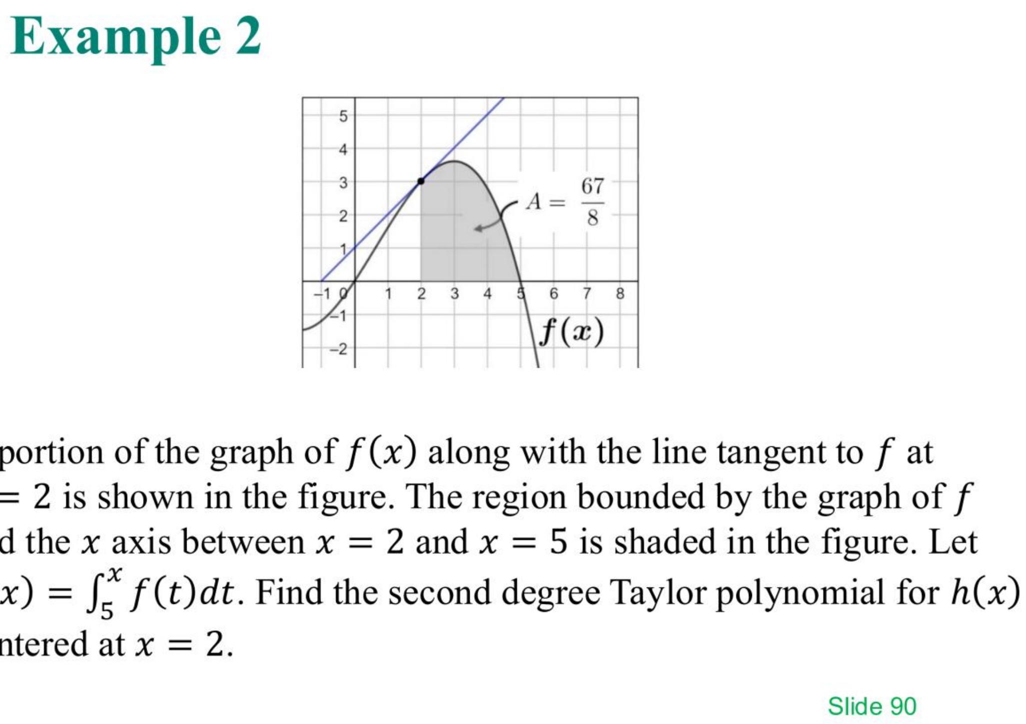
Write the equation of the **third-degree Taylor polynomial** for $f(x)$ about $x = 2$.

Key Points:

- Use the Taylor polynomial formula up to the third derivative term.
- Plug in the values from the table.
- The polynomial will be: $P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$.

Why It Matters:

This is the most common type of question — using a table of derivatives to quickly build a Taylor polynomial without calculating derivatives yourself.



Example 2

A portion of the graph of $f(x)$ along with the line tangent to f at $x = 2$ is shown in the figure. The region bounded by the graph of f and the x axis between $x = 2$ and $x = 5$ is shaded in the figure. Let $h(x) = \int_5^x f(t)dt$. Find the second degree Taylor polynomial for $h(x)$ centered at $x = 2$.

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Title: Example 2 (Graph + Integral)

Clear Explanation:

A graph of f(x) is shown with the tangent line at x = 2 . The shaded region is between x = 2 and x = 5 .

Define h(x) = \int_{5}^{x} f(t) \, dt .

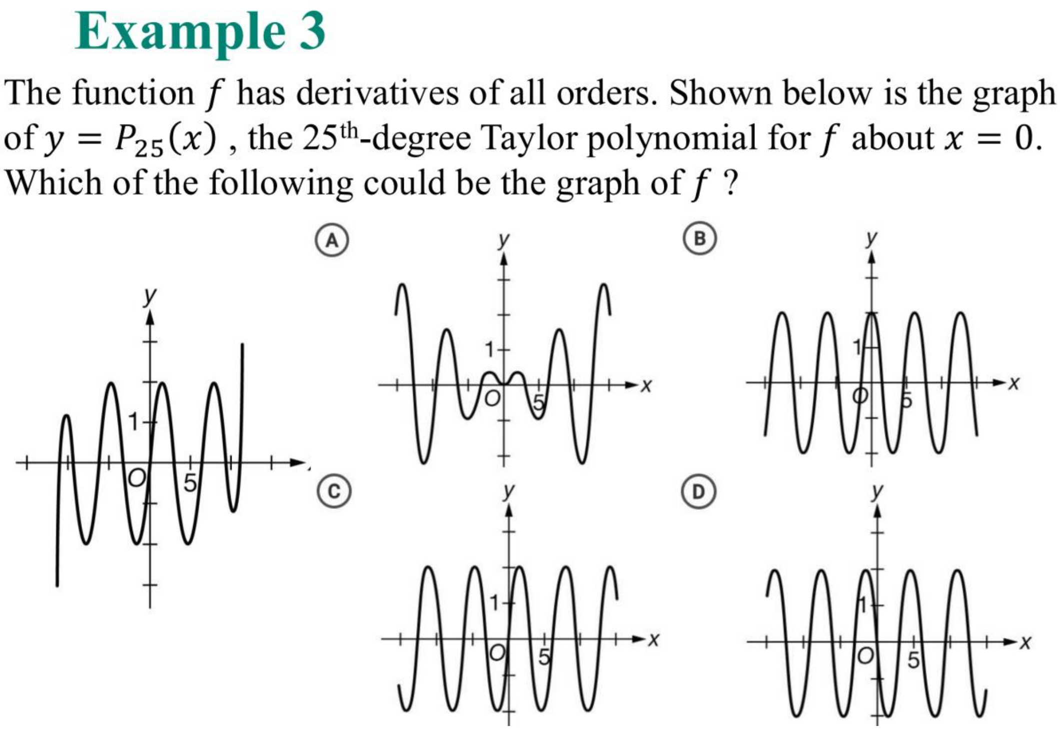
Find the **second-degree Taylor polynomial** for h(x) centered at x = 2 .

Key Points:

- Use the Fundamental Theorem of Calculus: h'(x) = f(x) , h''(x) = f'(x) , etc.
- h(2) = -\int_{2}^{5} f(t) \, dt (area under the curve).
- Build the polynomial using values at x = 2 .

Why It Matters:

This combines integration with Taylor polynomials — a very common AP-style problem.



Example 3

The function f has derivatives of all orders. Shown below is the graph of $y = P_{25}(x)$, the 25th-degree Taylor polynomial for f about $x = 0$. Which of the following could be the graph of f ?

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Title: Example 3 (Graph of P_{25}(x))

Clear Explanation:

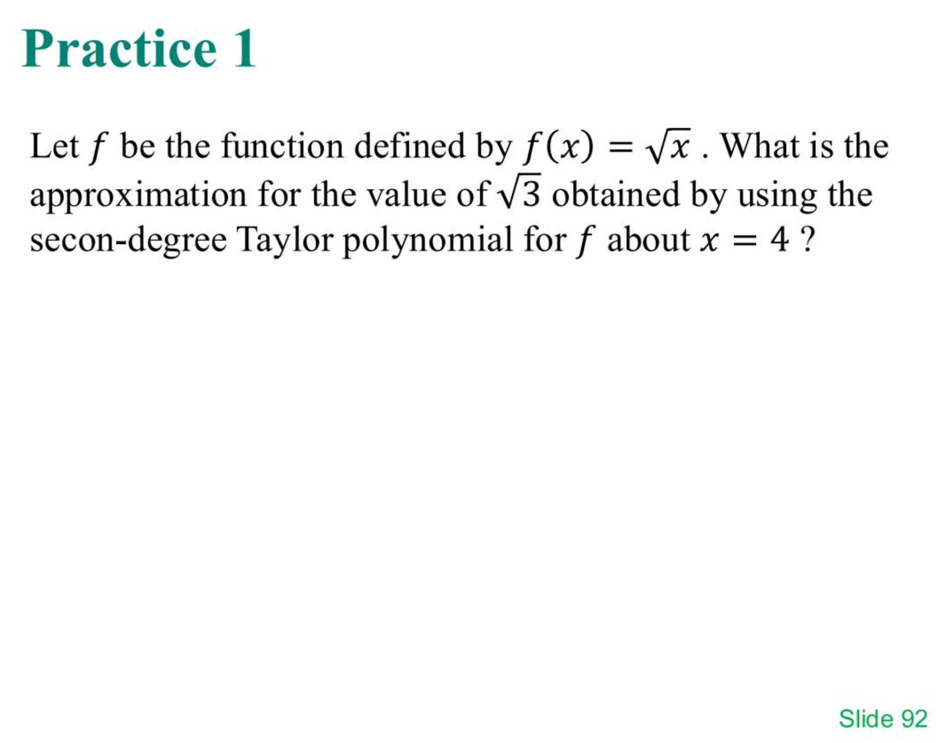
The graph shown is y = P_{25}(x) , the 25th-degree Taylor polynomial for f(x) about x = 0 .
Which of the four graphs (A, B, C, or D) could be the actual graph of the original function f(x) ?

Key Points:

- A high-degree Taylor polynomial closely follows the original function near the center (x = 0).
- Look for the graph that matches the polynomial's behavior near x = 0 but may differ far away.

Why It Matters:

This tests your understanding of how well Taylor polynomials approximate the real function.



Practice 1

Let f be the function defined by $f(x) = \sqrt{x}$. What is the approximation for the value of $\sqrt{3}$ obtained by using the secon-degree Taylor polynomial for f about $x = 4$?

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Title: Practice 1

Clear Explanation:

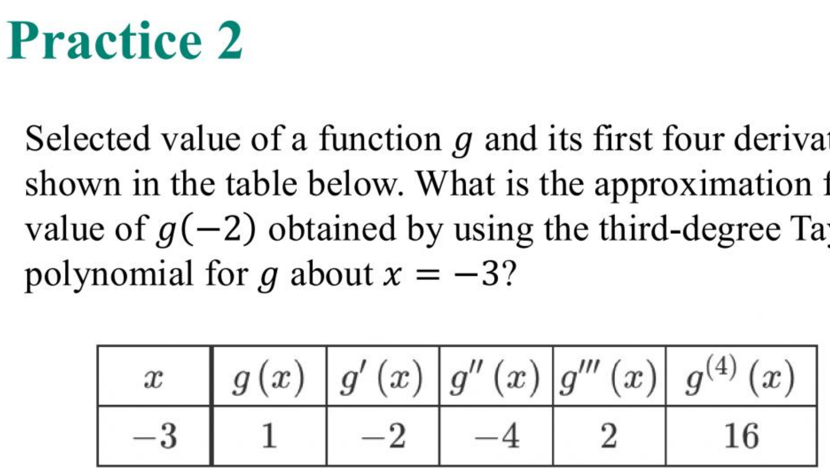
Let f(x) = \sqrt{x} . What is the approximation for \sqrt{3} obtained by using the **second-degree Taylor polynomial** for f(x) about x = 4 ?

Key Points:

- Center at a = 4 (because \sqrt{4} = 2 is easy).
- Compute f(4) , f'(4) , f''(4) .
- Plug into the Taylor polynomial formula.

Why It Matters:

Classic approximation problem using Taylor polynomials for square roots.



Practice 2

Selected value of a function g and its first four derivatives are shown in the table below. What is the approximation for the value of $g(-2)$ obtained by using the third-degree Taylor polynomial for g about $x = -3$?

x	$g(x)$	$g'(x)$	$g''(x)$	$g'''(x)$	$g^{(4)}(x)$
-3	1	-2	-4	2	16

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Title: Practice 2 (Table of derivatives)

Clear Explanation:

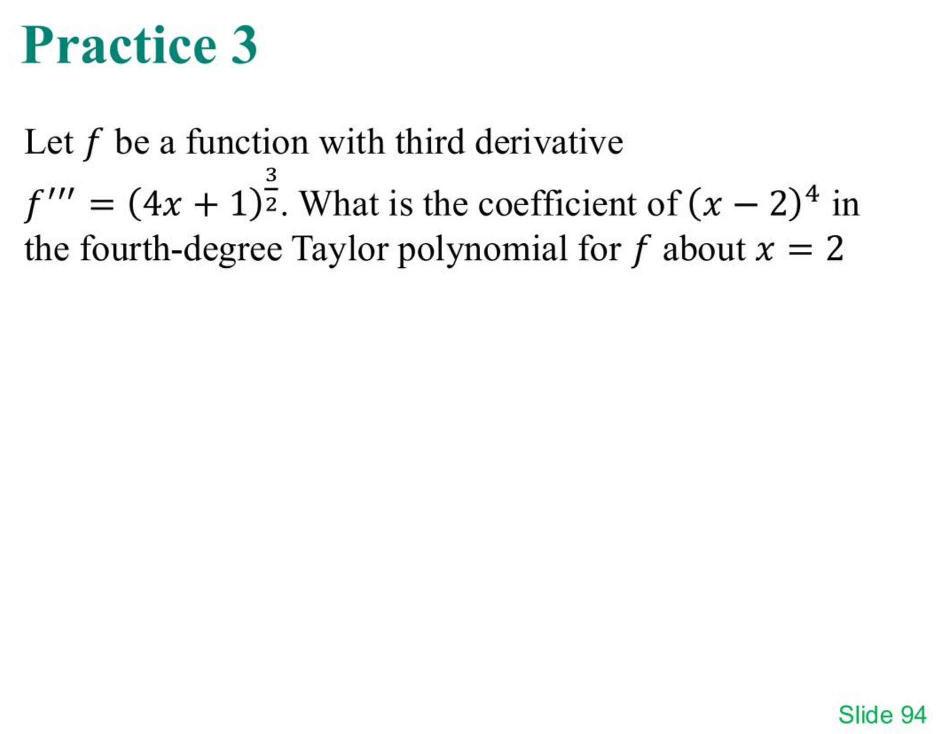
Selected values of a function g and its first four derivatives at x = -3 are given in a table.
What is the approximation for g(-2) using the **third-degree Taylor polynomial** for g about x = -3 ?

Key Points:

- Use the table values directly.
- Center is x = -3 , evaluate at x = -2 .

Why It Matters:

Practice using tables to build and evaluate Taylor polynomials quickly.



Practice 3

Let f be a function with third derivative $f''' = (4x+1)^{\frac{3}{2}}$. What is the coefficient of $(x-2)^4$ in the fourth-degree Taylor polynomial for f about $x = 2$

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Title: Practice 3

Clear Explanation:

Let f be a function with third derivative f'''(x) = (4x + 1)^2 .

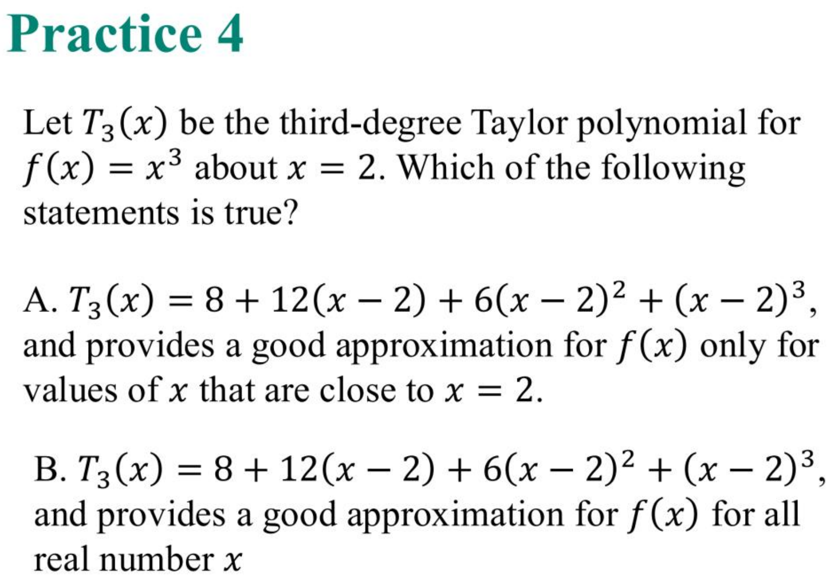
What is the coefficient of (x - 2)^4 in the **fourth-degree Taylor polynomial** for f about x = 2 ?

Key Points:

- The coefficient of the (x - a)^n term is \frac{f^{(n)}(a)}{n!} .
- Here n = 4, a = 2, so you need the fourth derivative at x = 2.

Why It Matters:

Tests whether you understand the general term of the Taylor polynomial.



Practice 4

Let $T_3(x)$ be the third-degree Taylor polynomial for $f(x) = x^3$ about $x = 2$. Which of the following statements is true?

A. $T_3(x) = 8 + 12(x-2) + 6(x-2)^2 + (x-2)^3$, and provides a good approximation for $f(x)$ only for values of x that are close to $x = 2$.

B. $T_3(x) = 8 + 12(x-2) + 6(x-2)^2 + (x-2)^3$, and provides a good approximation for $f(x)$ for all real number x

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Title: Practice 4

Clear Explanation:

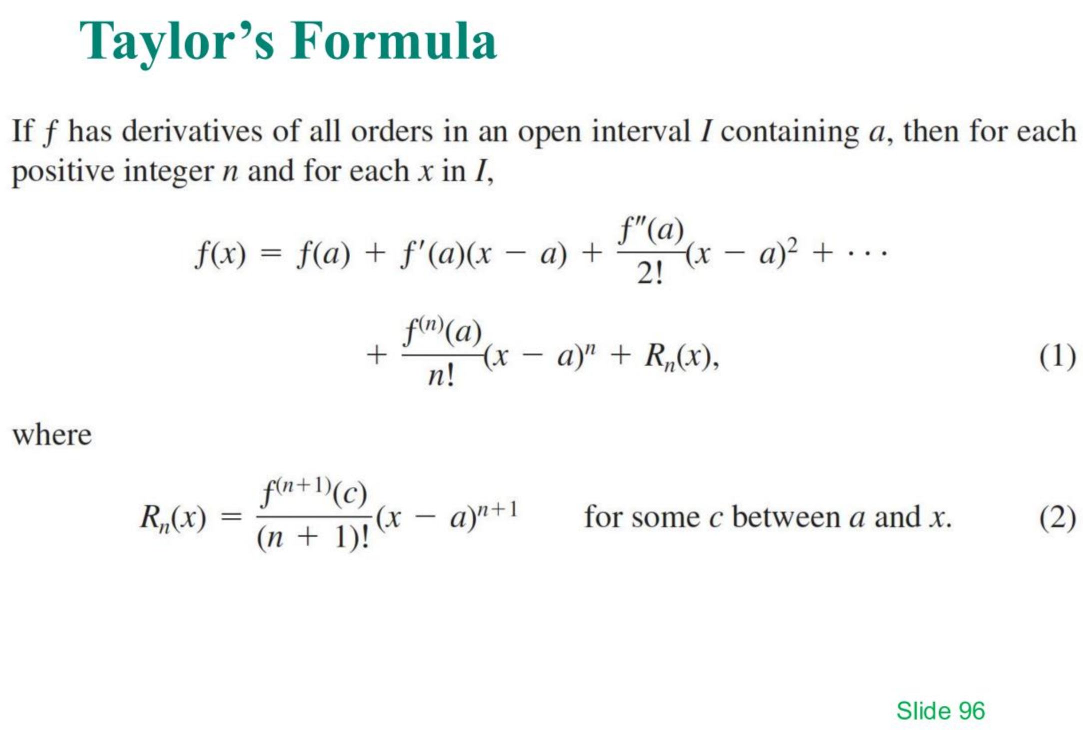
Let T_3(x) be the third-degree Taylor polynomial for f(x) = x^3 about x = 2 .
Which statement is true?
A. It approximates f(x) well only near x = 2 .
B. It approximates f(x) well for all real x .

Key Points:

- For polynomials, the Taylor polynomial of degree ≥ the degree of the function is **exactly** the function itself.
- Here f(x) = x^3 is already degree 3, so T_3(x) = x^3 exactly.

Why It Matters:

Important reminder: for polynomials, Taylor polynomials of sufficient degree are exact (no error).



Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \qquad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c \text{ between } a \text{ and } x. \qquad (2)$$

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Title: Taylor's Formula (with Remainder)

Clear Explanation:

Taylor's Formula gives the exact relationship between a function and its Taylor polynomial:

f(x) = P_n(x) + R_n(x)

where R_n(x) is the **remainder** (error) term, given by Lagrange form:

R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}

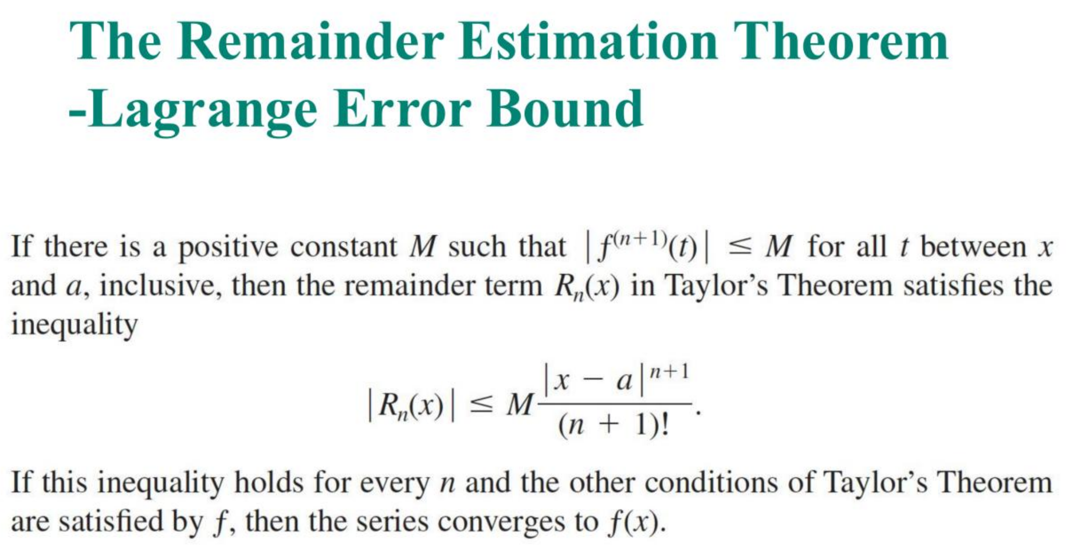
for some c between a and x .

Key Points:

- P_n(x) = Taylor polynomial
- R_n(x) = remainder (error)
- Lagrange form is the most commonly used.

Why It Matters:

This is the theoretical foundation for error analysis in Taylor approximations.



The Remainder Estimation Theorem -Lagrange Error Bound

If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

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Title: The Remainder Estimation Theorem (Lagrange Error Bound)

Clear Explanation:

If $|f^{(n+1)}(t)| \leq M$ for all t between a and x, then the absolute error satisfies:

$|R_n(x)| \leq M \frac{|x - a|^{n+1}}{(n+1)!}$

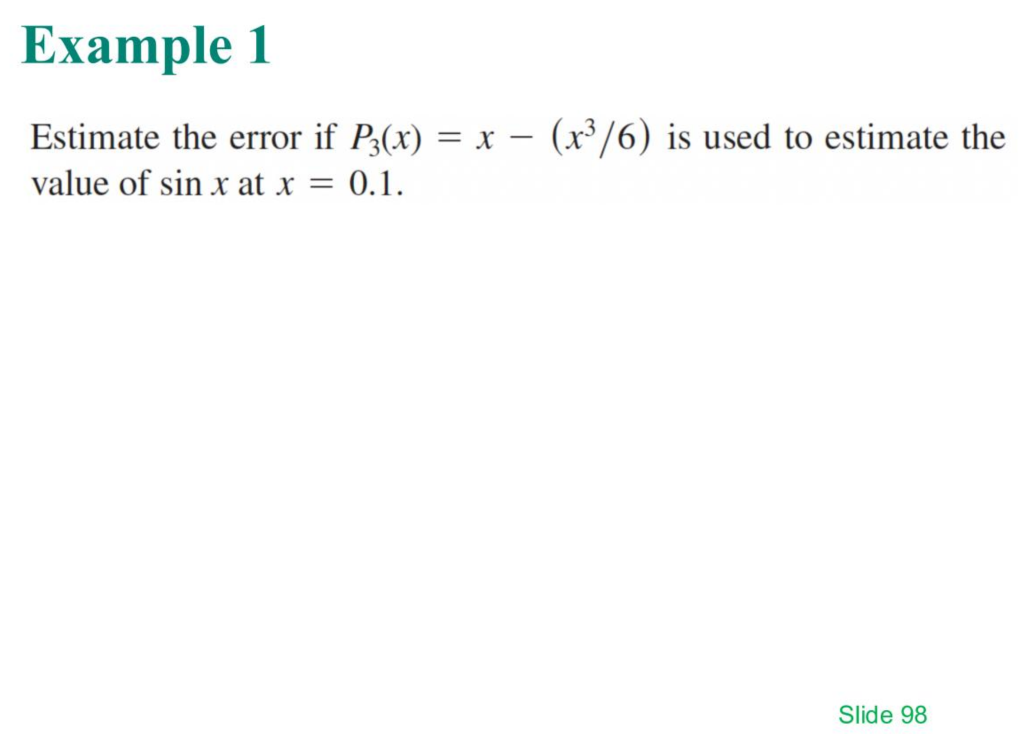
This is called the **Lagrange Error Bound**.

Key Points:

- Gives an upper bound on the maximum possible error.
- Very useful for AP exam questions.

Why It Matters:

Allows you to guarantee how accurate your approximation is without knowing the exact function.



Example 1

Estimate the error if $P_3(x) = x - (x^3/6)$ is used to estimate the value of $\sin x$ at $x = 0.1$.

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Title: Example 1 (Error estimation for sin x)

Clear Explanation:

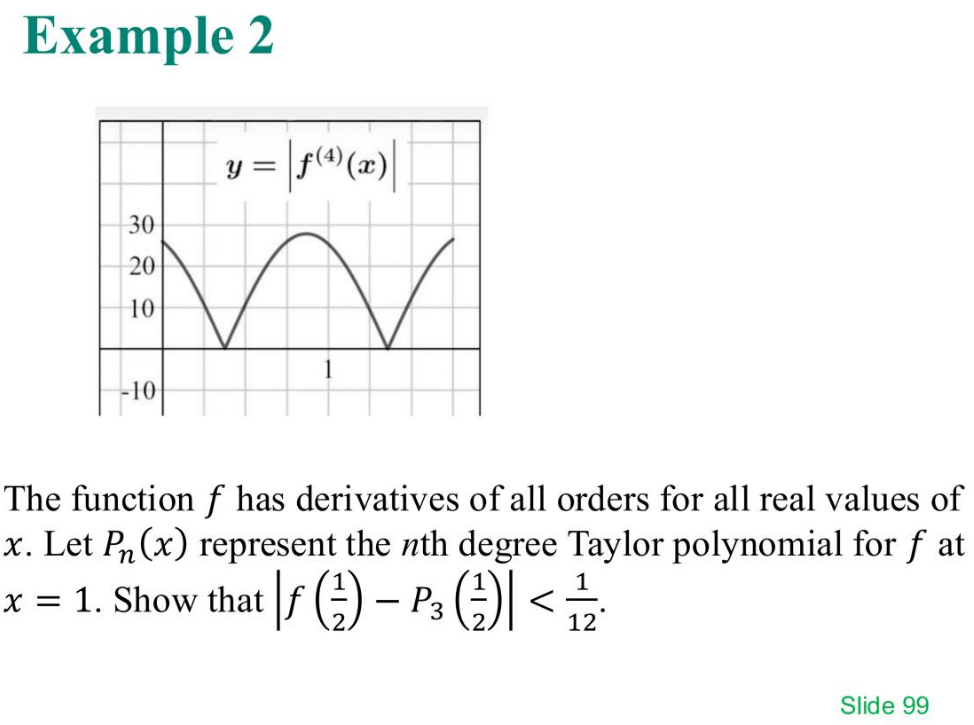
Estimate the error if the third-degree Taylor polynomial $P_3(x) = x - \frac{x^3}{6}$ is used to approximate $\sin x$ at $x = 0.1$.

Key Points:

- Use the Lagrange Error Bound with $n = 3$, $a = 0$.
- Need the maximum of $|f^{(4)}(t)| = |\cos t| \leq 1$.

Why It Matters:

Classic error-bound calculation using the sine series.



Example 2

The function f has derivatives of all orders for all real values of x. Let $P_n(x)$ represent the nth degree Taylor polynomial for f at $x = 1$. Show that $\left|f\left(\frac{1}{2}\right) - P_3\left(\frac{1}{2}\right)\right| < \frac{1}{12}$.

Page 20 Annotation

Title: Example 2 (Error bound with graph)

Clear Explanation:

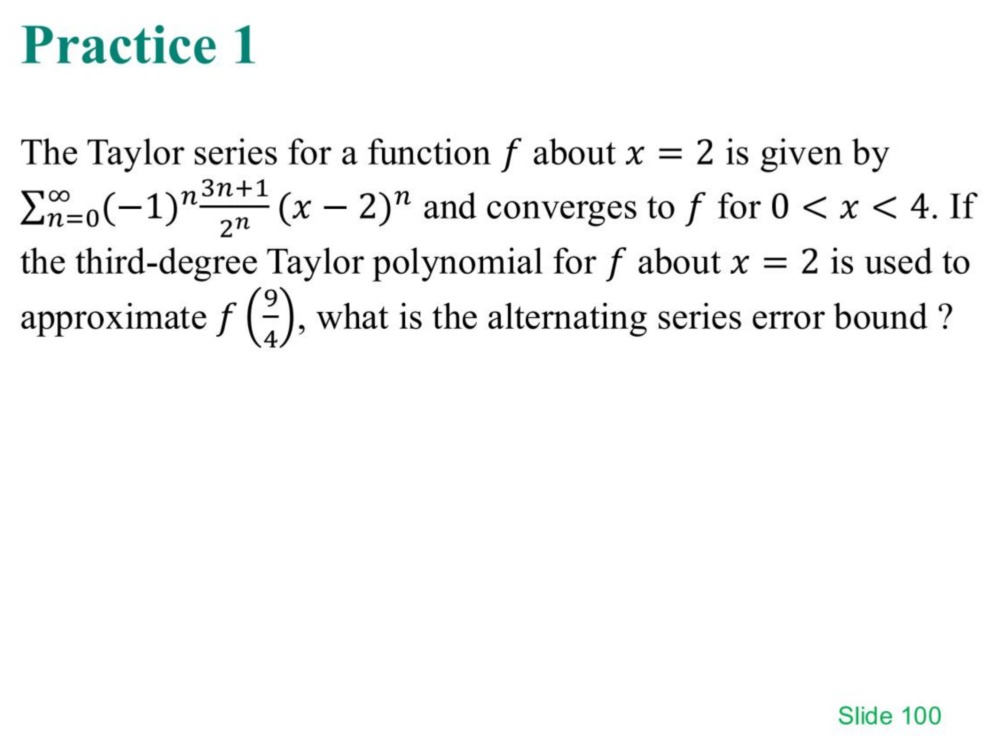
The function f has derivatives of all orders. P_n(x) is the nth-degree Taylor polynomial about x = 1 .
Show that |f(1/2) - P_3(1/2)| < \frac{1}{12} .

Key Points:

- Use the graph of |f^{(4)}(x)| to find M.
- Apply the error bound formula.

Why It Matters:

Combines graphical information with the error bound theorem.



Practice 1

The Taylor series for a function f about $x = 2$ is given by $\sum_{n=0}^{\infty}(-1)^n\frac{3n+1}{2^n}(x-2)^n$ and converges to f for $0 < x < 4$. If the third-degree Taylor polynomial for f about $x = 2$ is used to approximate $f\left(\frac{9}{4}\right)$, what is the alternating series error bound ?

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Title: Practice 1 (Alternating series error bound)

Clear Explanation:

The Taylor series for a function about $x = 2$ is given and converges for $0 < x < 4$.

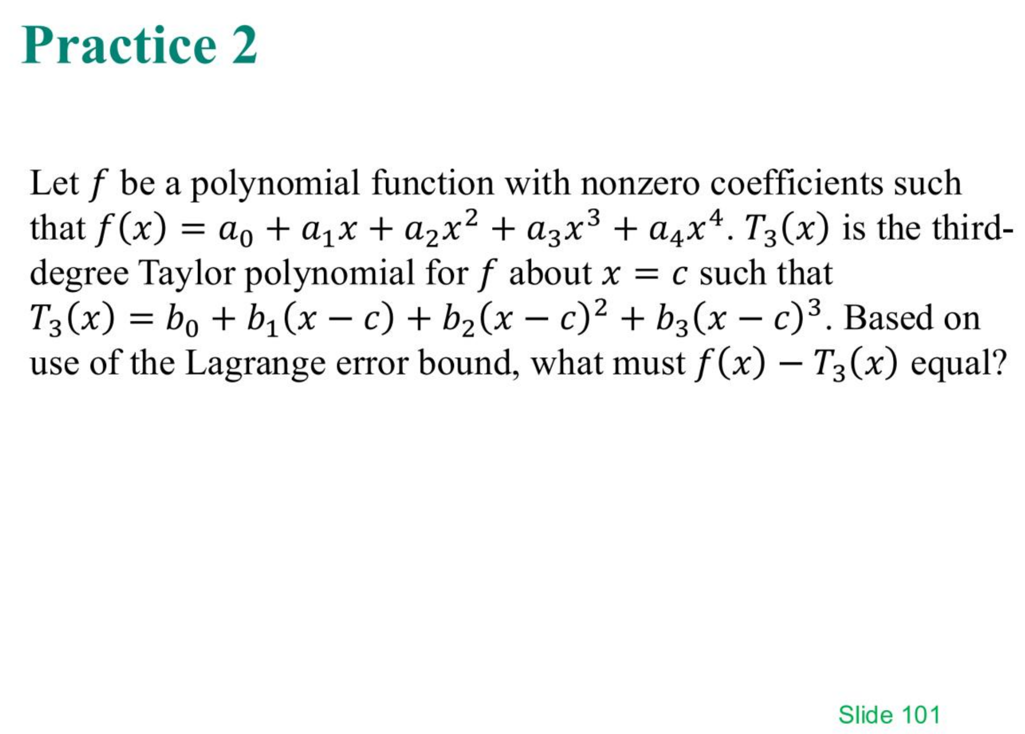
If the third-degree Taylor polynomial is used to approximate f(9/4), what is the alternating series error bound?

Key Points:

- For alternating series, the error is less than the first omitted term.
- Use the next term in the series.

Why It Matters:

Shows the difference between Lagrange error bound and alternating series error bound.



Practice 2

Let f be a polynomial function with nonzero coefficients such that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$. $T_3(x)$ is the third-degree Taylor polynomial for f about $x = c$ such that $T_3(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3$. Based on use of the Lagrange error bound, what must $f(x) - T_3(x)$ equal?

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Title: Practice 2 (Lagrange Error Bound for a polynomial)

Clear Explanation:

This practice problem asks you to connect a third-degree Taylor polynomial with the Lagrange error bound. The function f(x) is a fourth-degree polynomial, while T_3(x) keeps terms only up to degree 3 around x=c . Therefore, the difference f(x)-T_3(x) is the remainder term, which comes from the fourth derivative.

For a fourth-degree polynomial, the Lagrange remainder after the third-degree Taylor polynomial is exactly the fourth-degree part of the function. In Lagrange form, it is

R_3(x)=

rac{f^{(4)}(z)}{4!}(x-c)^4

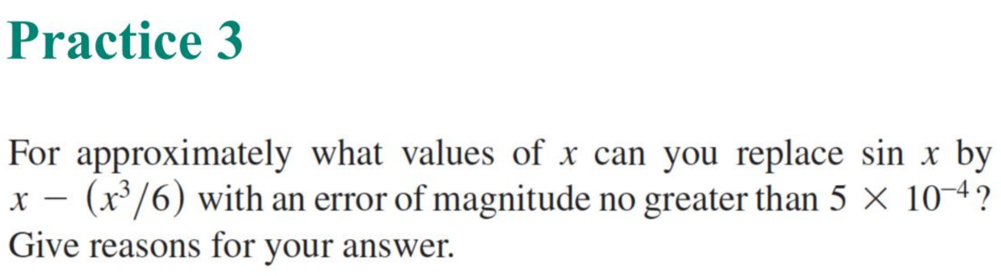
for some z between x and c . Since f^{(4)}(x)=24a_4 , the remainder equals a_4(x-c)^4 .

Key Points:

- T_3(x) includes terms through the third derivative.
- The error f(x)-T_3(x) is the fourth-order remainder.
- For a fourth-degree polynomial, this remainder is exact, not just an estimate.
- The answer should be written as a fourth-power term involving (x-c)^4.

Why It Matters:

This problem shows that the Lagrange error bound is not only for approximating error. For polynomials, it can identify the exact missing term after a Taylor polynomial is truncated.



Practice 3

For approximately what values of x can you replace $\sin x$ by $x - (x^3/6)$ with an error of magnitude no greater than 5×10^{-4}? Give reasons for your answer.

Slide 102

Page 23 Annotation

Title: Practice 3 (Error bound for approximating \sin x)

Clear Explanation:

This practice problem asks when \sin x can be replaced by its third-degree Maclaurin polynomial x-
rac{x^3}{6} with an error no greater than 5 imes10^{-4}.
The next nonzero term in the sine series is
rac{x^5}{5!}. For an alternating series like \sin x, the error after stopping at x-
rac{x^3}{6} is at most the absolute value of the first omitted term:
|R| \leq
rac{|x|^5}{5!}.
So we need

rac{|x|^5}{120} \leq 5 imes10^{-4}.
Solving this inequality gives the approximate range of x-values where the approximation is accurate enough.

Key Points:

- x-
rac{x^3}{6} is the third-degree Maclaurin polynomial for \sin x.
- Use the alternating series error bound.
- The first omitted term is
rac{x^5}{5!}.
- Set
rac{|x|^5}{120} \leq 5 imes10^{-4} and solve for |x|.

Why It Matters:

This is a classic example of using Taylor series error bounds to decide when a polynomial approximation is reliable.